# Introduction to QPA

## Part 2

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Third GAP Days

# Outline

# Nakayama algebras

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} n$$

or

$$\begin{tikzcd}
& 1 \arrow[r, "a_1"] & 2 \arrow[dr, "a_2"] & \\
n \arrow[ur, "a_n"] & & & 3 \arrow[dl, "a_3"] \\
& \arrow[ul, "a^{n-1}"] & \cdots\cdots &
\end{tikzcd}$$

# A Nakayama algebra

$$A = kQ/\langle\rho\rangle \qquad Q\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \qquad \rho = \{ab\}$$

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Indecomposable projective $A$-modules:

$$P1\colon\ k \xrightarrow{1} k \longrightarrow 0 \longrightarrow 0$$

$$P2\colon\ 0 \longrightarrow k \xrightarrow{1} k \xrightarrow{1} k$$

$$P3\colon\ 0 \longrightarrow 0 \longrightarrow k \xrightarrow{1} k$$

$$P4\colon\ 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow k$$

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```
gap> NakayamaAlgebra([2,3,2,1], Rationals);
```

# Truncated path algebras

- $kQ/I$, where $I$ generated by all paths of length $n$

```
gap> Q := Quiver(3, [[1,2,"a"],
                     [2,1,"b"],
                     [2,2,"c"]]);;
gap> A := TruncatedPathAlgebra(Rationals, Q, 3);
```

# Recall: Modules (representations) in QPA

$$Q\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

$$M\colon\ k \xrightarrow{(2\ 0)} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k$$

```
gap> Q := Quiver(3, [[1,2,"a"],[2,3,"b"]]);;
gap> kQ := PathAlgebra(Rationals, Q);;
gap> M := RightModuleOverPathAlgebra
          (kQ, [1,2,1],
           [["a", [[2,0]]], ["b", [[4],[-1]]]]);
<[ 1, 2, 1 ]>
```

# Module attributes

$$M\colon\ k^1 \xrightarrow{(2\ 0)} k^2 \xrightarrow{\left(\begin{smallmatrix}4\\-1\end{smallmatrix}\right)} k^1$$

- `RightActingAlgebra`: $kQ$
- `LeftActingDomain`: $k$
- `DimensionVector`: $(1, 2, 1)$
- `MatricesOfPathAlgebraModule`: $\left(\begin{pmatrix}2 & 0\end{pmatrix}, \begin{pmatrix}4\\-1\end{pmatrix}\right)$
- `Dimension`: $4 = 1 + 2 + 1$

# Module attributes

$$M\colon\; k^1 \xrightarrow{(2\ 0)} k^2 \xrightarrow{\left(\begin{smallmatrix}4\\-1\end{smallmatrix}\right)} k^2$$

- `Basis`:

$$\begin{aligned}1 &\to (0,0) \to 0\\ 0 &\to (1,0) \to 0\\ 0 &\to (0,1) \to 0\\ 0 &\to (0,0) \to 1\end{aligned}$$

- `MinimalGeneratingSetOfModule`:

$$\begin{aligned}1 &\to (0,0) \to 0\\ 0 &\to (0,0) \to 1\end{aligned}$$

# Submodules

$$N \overset{i}{\hookrightarrow} M$$

- Categorical view of submodules
- A submodule is given by an inclusion homomorphism
- A submodule is not a subset

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- Categorical view of submodules
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- A submodule is not a subset
- `SubRepresentation`: $N$
- `SubRepresentationInclusion`: $i$

# Direct sum

$$\begin{tikzcd}
M_1 \arrow[dr, hook, "i_1"] & & M_2 \\
M_2 \arrow[r, hook, "i_2"] & M_1 \oplus M_2 \oplus M_3 \arrow[ur, two heads, "p_1"] \arrow[r, two heads, "p_2"] \arrow[dr, two heads, "p_3"] & M_2 \\
M_3 \arrow[ur, hook, "i_3"] & & M_3
\end{tikzcd}$$

- `DirectSumOfQPAModules`: $M_1 \oplus M_2 \oplus M_3$
- `DirectSumInclusions`: $(i_1, i_2, i_3)$
- `DirectSumProjections`: $(p_1, p_2, p_3)$

# Radical, socle and top

$$\begin{tikzcd} \operatorname{rad} M \arrow[r, hook, "i"] & M \arrow[r, two heads, "p"] & \operatorname{top} M \\ & \operatorname{soc} M \arrow[u, hook, "j"] & \end{tikzcd}$$

- `RadicalOfModule`: $\operatorname{rad} M$
- `RadicalOfModuleInclusion`: $i$
- `SocleOfModule`: $\operatorname{soc} M$
- `SocleOfModuleInclusion`: $j$
- `TopOfModule`: $\operatorname{top} M$
- `TopOfModuleInclusion`: $p$

# Modules: equality and isomorphism

Three ways to compare modules $M$ and $N$:

- `IsIdenticalObj(M, N)`
- `M = N`
- `IsomorphicModules(M, N)`

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For isomorphic modules:

- `IsomorphismOfModules(M, N)` produces isomorphism $M \xrightarrow{\cong} N$

# Simple modules

$$Q\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Simple $kQ$-modules:

$$S_1:\ k \longrightarrow 0 \longrightarrow 0$$
$$S_2:\ 0 \longrightarrow k \longrightarrow 0$$
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In QPA: `SimpleModules` gives $(S_1, S_2, S_3)$

# Indecomposable projective modules

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Indecomposable projective $kQ$-modules:

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# Indecomposable injective modules

$$Q\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Indecomposable injective $kQ$-modules:

$$I_1:\ k \longrightarrow 0 \longrightarrow 0$$

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In QPA: `IndecInjectiveModules` gives $(I_1, I_2, I_3)$

# Homomorphisms

Recall:

$$\begin{tikzcd}
M: \arrow[d, "h"] & 0 \arrow[r] \arrow[d] & k \arrow[r, "5"] \arrow[d, "(3\ 2)"'] & k \arrow[d, "1"] \\
N: & k \arrow[r, "(0\ 3)"'] & k^2 \arrow[r, "\begin{pmatrix} 1 \\ 1 \end{pmatrix}"] & k
\end{tikzcd}$$

```
gap> h := RightModuleHomOverAlgebra
          (M, N, [ [[0]], [[3,2]], [[1]] ]);
<<[ 0, 1, 1 ]> ---> <[ 1, 2, 1 ]>>
```

# Hom spaces

- `HomOverAlgebra(M,N)` gives $k$-basis for $\mathrm{Hom}_A(M, N)$.
- $k$-structure on homomorphisms in $\mathrm{Hom}_A(M, N)$:
  use `f+g` and `scalar*f`

# Composition of homomorphisms

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

Composition: `f*g`

# Kernel, Cokernel, Image

$$M \xrightarrow{f} N$$

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$$\ker f \xhookrightarrow{i} M \xrightarrow{f} N$$

- `Kernel`: $\ker f$
- `KernelInclusion`: $i$

# Kernel, Cokernel, Image

$$\ker f \xhookrightarrow{i} M \xrightarrow{f} N \xrightarrow{p} \mathrm{coker}\, f$$

- `Kernel`: $\ker f$
- `KernelInclusion`: $i$
- `CoKernel`: $\mathrm{coker}\, f$
- `CoKernelProjection`: $p$

# Kernel, Cokernel, Image

$$\begin{tikzcd} \ker f \arrow[r, hook, "i"] & M \arrow[r, "f"] & N \arrow[r, two heads, "p"] & \operatorname{coker} f \\ & & \operatorname{im} f \arrow[u, hook, "j"] & \end{tikzcd}$$

- `Kernel`: $\ker f$
- `KernelInclusion`: $i$
- `CoKernel`: $\operatorname{coker} f$
- `CoKernelProjection`: $p$
- `Image`: $\operatorname{im} f$
- `ImageInclusion`: $j$

# Chain complexes

$$C : \cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \to \cdots$$

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- To represent a chain complex: Need infinite list $(\ldots, d_2, d_1, d_0, d_{-1}, \ldots)$ of differentials.

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- To represent a chain complex: Need infinite list $(\ldots, d_2, d_1, d_0, d_{-1}, \ldots)$ of differentials.
- Can not store all the differentials.
- Need to describe them with finite data.

# Chain complexes in QPA

Divide complex in three parts:

$$\underbrace{\cdots \xrightarrow{d_{b+m+1}} \xrightarrow{d_{b+m}}}_{\substack{\text{“positive”} \\ \text{(infinite)}}} \underbrace{\xrightarrow{d_{b+m-1}} \cdots \xrightarrow{d_b}}_{\substack{\text{“middle”} \\ \text{(finite)}}} \underbrace{\xrightarrow{d_{b-1}} \xrightarrow{d_{b_2}} \cdots}_{\substack{\text{“negative”} \\ \text{(infinite)}}}$$

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- Middle part: List of differentials
- Positive/negative part: Three possibilities

# Possibilities for the infinite parts

Consider the positive part:

$$\cdots \xrightarrow{d_3} \xrightarrow{d_2} \xrightarrow{d_1}$$

(assuming it starts with $d_1$)

# Possibility 1: Repeating list

- The same list $(r_1, \ldots, r_n)$ of differentials repeated infinitely.

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$$\cdots \xrightarrow{d_9} \xrightarrow{d_8} \xrightarrow{d_7} \xrightarrow{d_6} \xrightarrow{d_5} \xrightarrow{d_4} \xrightarrow[r_3]{d_3} \xrightarrow[r_2]{d_2} \xrightarrow[r_1]{d_1}$$

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- Special case: Zero

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- Initial differential $d_1$
- Function $f$ producing $d_{i+1}$ from $d_i$

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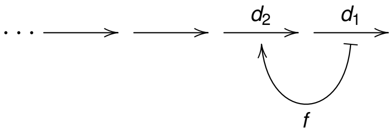

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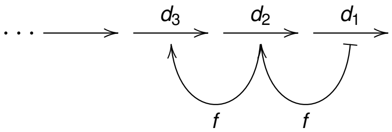

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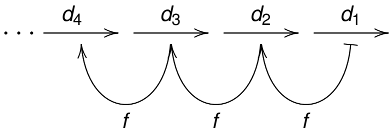

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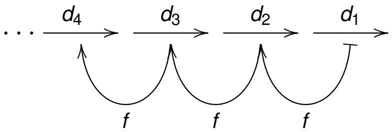


- Can convert to “repeating list” if repetition is detected

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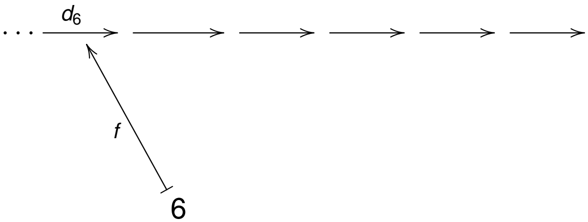

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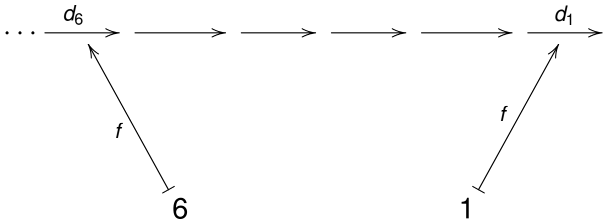

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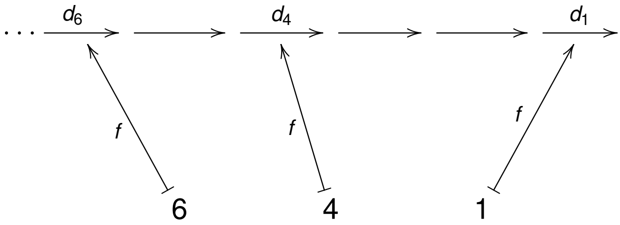

# Creating a chain complex

$$\underbrace{\cdots \xrightarrow{d_{b+m+1}} \xrightarrow{d_{b+m}}}_{\text{positive}} \underbrace{\xrightarrow{d_{b+m-1}} \cdots \xrightarrow{d_b}}_{\text{middle}} \underbrace{\xrightarrow{d_{b-1}} \xrightarrow{d_{b_2}} \cdots}_{\text{negative}}$$

Must specify:

- Position $b$
- Middle part: $(d_b, \ldots, d_{b+m-1})$
- Positive part: Repeating list or inductive function or positional function
- Negative part: Repeating list or inductive function or positional function

# Special complex constructors

- `ZeroComplex`
- `FiniteComplex`
- `StalkComplex`

# Projective resolutions

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

- `ProjectiveResolution`